

# A NOTE ON POLARIZATIONS OF FINITELY GENERATED FIELDS

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## INTRODUCTION

In the paper [5], we established Northcott's theorem for height functions over finitely generated fields. Unfortunately, Northcott's theorem on finitely generated fields does not hold in general (cf. Remark A.3). Actually, it depends on the choice of a polarization. In this short note, we will propose a weaker condition of the polarization to guarantee Northcott's theorem. We will also show the generalization of conjectures of Bogomolov and Lang in [6] under the weaker polarization.

First of all, let us introduce the weaker condition of a polarization. Let  $K$  be a finitely generated field over  $\mathbb{Q}$ . A polarization  $\overline{B} = (B; \overline{H}_1, \dots, \overline{H}_d)$  of  $K$  is said to be *fairly large* if there are generically finite morphisms  $\mu : B' \rightarrow B$  and  $\nu : B' \rightarrow (\mathbb{P}_{\mathbb{Z}}^1)^d$  of flat and projective integral schemes over  $\mathbb{Z}$ , and nef and big  $C^\infty$ -hermitian  $\mathbb{Q}$ -line bundles  $\overline{L}_1, \dots, \overline{L}_d$  on  $\mathbb{P}_{\mathbb{Z}}^1$  such that a positive power of  $\mu^*(\overline{H}_i) \otimes \nu^*(p_i^*(\overline{L}_i))^{\otimes -1}$  has a small section for every  $i$ , where  $p_i : (\mathbb{P}_{\mathbb{Z}}^1)^d \rightarrow \mathbb{P}_{\mathbb{Z}}^1$  is the projection to the  $i$ -th factor. Then, we have the following Theorem I, Theorem II and Theorem III, which are generalizations of the previous results.

**Theorem I** ([5, Theorem 4.3]). *We assume that the polarization  $\overline{B}$  is fairly large. Let  $X$  be a geometrically irreducible projective variety over  $K$ , and  $L$  an ample line bundle on  $X$ . Then, for any number  $M$  and any positive integer  $e$ , the set*

$$\{x \in X(\overline{K}) \mid h_L^{\overline{B}}(x) \leq M, \quad [K(x) : K] \leq e\}$$

*is finite. Moreover, it can be generalized to the height of cycles on a projective variety (cf. Theorem 3.1).*

**Theorem II** ([6, Theorem A]). *We assume that the polarization  $\overline{B}$  is fairly large. Let  $A$  be an abelian variety over  $K$ , and  $L$  a symmetric ample line bundle on  $A$ . Let*

$$\langle \cdot, \cdot \rangle_L^{\overline{B}} : A(\overline{K}) \times A(\overline{K}) \rightarrow \mathbb{R}$$

*be a paring given by*

$$\langle x, y \rangle_L^{\overline{B}} = \frac{1}{2} \left( \hat{h}_L^{\overline{B}}(x + y) - \hat{h}_L^{\overline{B}}(x) - \hat{h}_L^{\overline{B}}(y) \right).$$

*For  $x_1, \dots, x_l \in A(\overline{K})$ , we denote  $\det \left( \langle x_i, x_j \rangle_L^{\overline{B}} \right)$  by  $\delta_L^{\overline{B}}(x_1, \dots, x_l)$ .*

*Let  $\Gamma$  be a subgroup of finite rank in  $A(\overline{K})$ , and  $X$  a subvariety of  $A_{\overline{K}}$ . Fix a basis  $\{\gamma_1, \dots, \gamma_n\}$  of  $\Gamma \otimes \mathbb{Q}$ . If the set  $\{x \in X(\overline{K}) \mid \delta_L^{\overline{B}}(\gamma_1, \dots, \gamma_n, x) \leq \epsilon\}$  is Zariski dense in  $X$  for every positive number  $\epsilon$ , then  $X$  is a translation of an abelian subvariety of  $A_{\overline{K}}$  by an element of  $\Gamma_{div}$ , where  $\Gamma_{div} = \{x \in A(\overline{K}) \mid nx \in \Gamma \text{ for some positive integer } n\}$ .*

**Theorem III** ([7, Theorem 5.1]). *We assume that the polarization  $\overline{B}$  is fairly large. For a subvariety  $X$  of  $A_{\overline{K}}$ , the following are equivalent.*

- (1)  *$X$  is a translation of an abelian subvariety by a torsion point.*
- (2) *The set  $\{x \in X(\overline{K}) \mid \hat{h}_L^{\overline{B}}(x) \leq \epsilon\}$  is Zariski dense in  $X$  for every  $\epsilon > 0$ .*
- (3) *The canonical height of  $X$  with respect to  $L$  and  $\overline{B}$  is zero, i.e.,  $\hat{h}_L^{\overline{B}}(X) = 0$ .*

## 1. FAIRLY LARGE POLARIZATION OF A FINITELY GENERATED FIELD

Let  $K$  be a finitely generated field over  $\mathbb{Q}$  with  $d = \text{tr.deg}_{\mathbb{Q}}(K)$ , and let  $B$  be a flat and projective integral scheme over  $\mathbb{Z}$  such that  $K$  is the function field of  $B$ . Let  $\overline{L}$  be a  $C^\infty$ -hermitian  $\mathbb{Q}$ -line bundle on  $B$ . Here we fix several notations.

•**nef**: We say  $\overline{L}$  is *nef* if  $c_1(\overline{L})$  is a semipositive form on  $B(\mathbb{C})$  and, for all one-dimensional integral closed subschemes  $\Gamma$  of  $B$ ,  $\widehat{\deg}(\overline{L}|_\Gamma) \geq 0$ .

•**big**:  $\overline{L}$  is said to be *big* if  $\text{rk}_{\mathbb{Z}} H^0(B, L^{\otimes m}) = O(m^d)$ , and there is a non-zero section  $s$  of  $H^0(B, L^{\otimes n})$  with  $\|s\|_{\text{sup}} < 1$  for some positive integer  $n$ .

• **$\mathbb{Q}$ -effective**:  $\overline{L}$  is said to be  *$\mathbb{Q}$ -effective* if there is a positive integer  $n$  and a non-zero  $s \in H^0(B, L^{\otimes n})$  with  $\|s\|_{\text{sup}} \leq 1$ . If  $\overline{L}_1 \otimes \overline{L}_2^{\otimes -1}$  is  $\mathbb{Q}$ -effective for  $C^\infty$ -hermitian  $\mathbb{Q}$ -line bundles  $\overline{L}_1, \overline{L}_2$  on  $B$ , then we denote this by  $\overline{L}_1 \preceq \overline{L}_2$ .

•**polarization**: A collection  $\overline{B} = (B; \overline{H}_1, \dots, \overline{H}_d)$  of  $B$  and nef  $C^\infty$ -hermitian  $\mathbb{Q}$ -line bundles  $\overline{H}_1, \dots, \overline{H}_d$  on  $B$  is called a *polarization of  $K$* .

•**fairly large polarization**: A polarization  $\overline{B} = (B; \overline{H}_1, \dots, \overline{H}_d)$  is said to be *fairly large* if there are generically finite morphisms  $\mu : B' \rightarrow B$  and  $\nu : B' \rightarrow (\mathbb{P}_{\mathbb{Z}}^1)^d$  of flat and projective integral schemes over  $\mathbb{Z}$ , and nef and big  $C^\infty$ -hermitian  $\mathbb{Q}$ -line bundles  $\overline{L}_1, \dots, \overline{L}_d$  on  $\mathbb{P}_{\mathbb{Z}}^1$  such that  $\mu^*(\overline{H}_i) \preceq \nu^*(p_i^*(\overline{L}_i))$  for all  $i$ , where  $p_i : (\mathbb{P}_{\mathbb{Z}}^1)^d \rightarrow \mathbb{P}_{\mathbb{Z}}^1$  is the projection to the  $i$ -th factor.

Finally we would like to give a simple sufficient condition for the fair largeness of a polarization. Let  $k$  be a number field, and  $O_k$  the ring of integer in  $k$ . Let  $B_1, \dots, B_l$  be projective and flat integral schemes over  $O_k$  whose generic fibers over  $O_k$  are geometrically irreducible. Let  $K_i$  be the function field of  $B_i$  and  $d_i$  the transcendence degree of  $K_i$  over  $k$ . We set  $B = B_1 \times_{O_k} \dots \times_{O_k} B_l$  and  $d = d_1 + \dots + d_l$ . Then, the function field of  $B$  is the quotient field of  $K_1 \otimes_k K_2 \otimes_k \dots \otimes_k K_l$ , which is denoted by  $K$ , and the transcendence degree of  $K$  over  $k$  is  $d$ . For each  $i$  ( $i = 1, \dots, l$ ), let  $\overline{H}_{i,1}, \dots, \overline{H}_{i,d_i}$  be nef and big  $C^\infty$ -hermitian  $\mathbb{Q}$ -line bundles on  $B_i$ . We denote by  $q_i$  the projection  $B \rightarrow B_i$  to the  $i$ -th factor. Then, we have the following.

**Proposition 1.1.** *A polarization  $\overline{B}$  of  $K$  given by*

$$\overline{B} = (B; q_1^*(\overline{H}_{1,1}), \dots, q_1^*(\overline{H}_{1,d_1}), \dots, q_l^*(\overline{H}_{l,1}), \dots, q_l^*(\overline{H}_{l,d_l}))$$

*is fairly large.*

*Proof.* Since there is a dominant rational map  $B_i \dashrightarrow (\mathbb{P}_{\mathbb{Z}}^1)^{d_i}$  by virtue of Noether's normalization theorem, we can find a birational morphism  $\mu_i : B'_i \rightarrow B_i$  of projective integral schemes over  $O_k$  and a generically finite morphism  $\nu_i : B'_i \rightarrow (\mathbb{P}_{\mathbb{Z}}^1)^{d_i}$ . We set  $B' = B'_1 \times_{O_k} \dots \times_{O_k} B'_l$ ,  $\mu = \mu_1 \times \dots \times \mu_l$  and  $\nu = \nu_1 \times \dots \times \nu_l$ . Let  $\overline{L}$  be a  $C^\infty$ -hermitian line bundle on  $\mathbb{P}_{\mathbb{Z}}^1$  given by  $(\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^1}(1), \|\cdot\|_{FS})$ . Note that  $\overline{L}$  is nef and big. Then, since  $\mu_i^*(\overline{H}_{i,j})$  is big, there is a positive integer

$a_{i,j}$  with  $\mu_i^*(\overline{H}_{i,j})^{\otimes a_{i,j}} \succsim \nu_i^*(p_j^*(\overline{L}))$ , that is,  $\mu_i^*(\overline{H}_{i,j}) \succsim \nu_i^*(p_j^*(\overline{L}^{\otimes 1/a_{i,j}}))$ . Thus, we get our proposition.  $\square$

## 2. COMPARISON OF HEIGHTS WITH RESPECT TO DIFFERENT POLARIZATIONS

First of all, let us recall the definition of height functions over finitely generated fields.

Let  $K$  be a finitely generated field over  $\mathbb{Q}$  with  $d = \text{tr. deg}_{\mathbb{Q}}(K)$ , and let  $\overline{B} = (B; \overline{H}_1, \dots, \overline{H}_d)$  be a polarization of  $K$ . First of all, let us recall the definition of height of  $\overline{K}$ -valued points. Let  $X$  be a geometrically irreducible projective variety over  $K$  and  $L$  an ample line bundle on  $X$ . Let us take a projective integral scheme  $\mathcal{X}$  over  $B$  and a  $C^\infty$ -hermitian  $\mathbb{Q}$ -line bundle  $\overline{\mathcal{L}}$  on  $\mathcal{X}$  such that  $X$  is the generic fiber of  $\mathcal{X} \rightarrow B$  and  $L$  is equal to  $\mathcal{L}_K$  in  $\text{Pic}(X) \otimes \mathbb{Q}$ . Then, for  $x \in X(\overline{K})$ , we define  $h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}}(x)$  to be

$$h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}}(x) = \frac{\widehat{\deg} \left( \widehat{c}_1(\overline{\mathcal{L}}) \cdot \prod_{j=1}^d \widehat{c}_1(\pi^*(\overline{H}_j)) \cdot \Delta_x \right)}{[K(x) : K]},$$

where  $\Delta_x$  is the Zariski closure in  $\mathcal{X}$  of the image  $\text{Spec}(\overline{K}) \rightarrow X \hookrightarrow \mathcal{X}$ , and  $\pi : \mathcal{X} \rightarrow B$  is the canonical morphism. By virtue of [5], if  $(\mathcal{X}', \mathcal{L}')$  is another model of  $(X, L)$  over  $B$ , then there is a constant  $C$  with  $|h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}}(x) - h_{(\mathcal{X}', \overline{\mathcal{L}'})}^{\overline{B}}(x)| \leq C$  for all  $x \in X(\overline{K})$ . Hence, we have the unique height function  $h_L^{\overline{B}}$  modulo the set of bounded functions.

More generally, we can define the height of cycles on  $X_{\overline{K}}$ . We assume that  $\overline{\mathcal{L}}$  is nef with respect to  $\pi : \mathcal{X} \rightarrow B$ , that is,

- (1) For any analytic maps  $h : M \rightarrow \mathcal{X}(\mathbb{C})$  from a complex manifold  $M$  to  $\mathcal{X}(\mathbb{C})$  with  $\pi(h(M))$  being a point,  $c_1(h^*(\overline{\mathcal{L}}))$  is semipositive.
- (2) For every  $b \in B$ , the restriction  $\mathcal{L}|_{X_b}$  of  $\mathcal{L}$  to the geometric fiber over  $b$  is nef.

Let  $Z$  be an effective cycle on  $X_{\overline{K}}$ . We assume that  $Z$  is defined over a finite extension field  $K'$  of  $K$ . Let  $B'$  be the normalization of  $B$  in  $K'$ , and let  $\rho : B' \rightarrow B$  be the induced morphism. Let  $\mathcal{X}'$  be the main component of  $\mathcal{X} \times_B B'$ . We set the induced morphisms as follows.

$$\begin{array}{ccc} \mathcal{X} & \xleftarrow{\tau} & \mathcal{X}' \\ \pi \downarrow & & \downarrow \pi' \\ B & \xleftarrow{\rho} & B' \end{array}$$

Let  $\mathcal{Z}$  be the Zariski closure of  $Z$  in  $\mathcal{X}'$ . Then the height  $h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}}(Z)$  of  $Z$  with respect to  $(\mathcal{X}, \overline{\mathcal{L}})$  and  $\overline{B}$  is defined by

$$h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}}(Z) = \frac{\widehat{\deg} \left( \widehat{c}_1(\tau^*(\overline{\mathcal{L}}))^{\dim Z + 1} \cdot \prod_{j=1}^d \widehat{c}_1(\pi'^*(\rho^*(\overline{H}_j))) \cdot \mathcal{Z} \right)}{[K' : K](\dim Z + 1) \deg_L(Z)}.$$

Note that the above definition does not depend on the choice of  $K'$  by the projection formula. Let  $(\mathcal{Y}, \overline{\mathcal{M}})$  be another model of  $(X, L)$  over  $B$  such that  $\overline{\mathcal{M}}$  is nef with respect to  $\mathcal{Y} \rightarrow B$ . Then, there is a constant  $C$  such that

$$\left| h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}}(Z) - h_{(\mathcal{Y}, \overline{\mathcal{M}})}^{\overline{B}}(Z) \right| \leq C$$

for all effective cycles  $Z$  of  $X_{\overline{K}}$  (cf. [7, Proposition 2.1]). Thus, we may denote by  $h_L^{\overline{B}}$  the class of  $h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}}$  modulo the set of bounded functions. Moreover, we say  $h_L^{\overline{B}}$  is the height function associated with  $L$  and  $\overline{B}$ .

Let  $k$  be a number field and  $O_k$  the ring of integers in  $k$ . Fix a positive integer  $d$ . For each  $1 \leq i \leq d$ , let  $B_i$  be a flat and projective integral scheme over  $O_k$  whose generic fiber over  $O_k$  is a geometrically irreducible curve over  $k$ . Let  $\overline{M}_i$  be a nef and big hermitian  $\mathbb{Q}$ -line bundle on  $B_i$ . Moreover, let  $B$  be a flat and projective integral scheme over  $O_k$ , and  $\nu : B \rightarrow B_1 \times_{O_k} \cdots \times_{O_k} B_d$  a generically finite morphism. We denote the function field of  $B$  (resp.  $B_i$ ) by  $K$  (resp.  $K_i$ ). Note that  $\text{tr. deg}_k(K) = d$  and  $\text{tr. deg}_k(K_i) = 1$  for all  $i$ . We set  $\overline{H}_i = \nu^* p_i^*(\overline{M}_i)$  for each  $i$  and  $\overline{H} = \bigotimes_{i=1}^d \overline{H}_i$ , where  $p_i : B_1 \times_{O_k} \cdots \times_{O_k} B_d \rightarrow B_i$  is the projection to the  $i$ -th factor. Further, we set

$$\lambda_i = \exp \left( - \frac{\widehat{\deg}(\widehat{c}_1(\overline{M}_i)^2)}{[k : \mathbb{Q}] \deg((M_i)_k)} \right).$$

Here we consider several kinds of polarizations of  $K$  as follows:

$$\begin{cases} \overline{B}_0 = (B; \overline{H}, \dots, \overline{H}), \\ \overline{B}_1 = (B; \overline{H}_1, \dots, \overline{H}_d), \\ \overline{B}_{i,j} = (B; \overline{H}_1, \dots, \overline{H}_{j-1}, (\mathcal{O}_B, \lambda_i | \cdot |_{\text{can}}), \overline{H}_{j+1}, \dots, \overline{H}_d) \quad \text{for } i \neq j. \end{cases}$$

A key result of this note is the following.

**Proposition 2.1.** *Let  $X$  be a geometrically irreducible projective variety over  $K$ , and  $L$  an ample line bundle on  $X$ . Let  $(\mathcal{X}, \overline{\mathcal{L}})$  be a model of  $(X, L)$  over  $B$  such that  $\overline{\mathcal{L}}$  is nef with respect to  $\mathcal{X} \rightarrow B$ . Then, for all effective cycles  $Z$  on  $X_{\overline{K}}$ ,*

$$h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}_0}(Z) = d! h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}_1}(Z) + \frac{d!}{2} \sum_{i \neq j} h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}_{i,j}}(Z).$$

In particular, we can find a constant  $C$  such that

$$h_L^{\overline{B}_0}(Z) \leq C h_L^{\overline{B}_1}(Z) + O(1)$$

for all effective cycles  $Z$  on  $X_{\overline{K}}$ .

*Proof.* Let  $Z$  be an effective cycle on  $X_{\overline{K}}$ . We assume that  $Z$  is defined over a finite extension field  $K'$  of  $K$ . Let  $B'$  be the normalization of  $B$  in  $K'$ , and let  $\rho : B' \rightarrow B$  be the induced morphism. Let  $\mathcal{X}'$  be the main component of  $\mathcal{X} \times_B B'$ . We set the induced morphisms as follows.

$$\begin{array}{ccc} \mathcal{X} & \xleftarrow{\tau} & \mathcal{X}' \\ \pi \downarrow & & \downarrow \pi' \\ B & \xleftarrow{\rho} & B' \end{array}$$

Let  $\mathcal{Z}$  be the Zariski closure of  $Z$  in  $\mathcal{X}'$ . We set  $\overline{D} = \nu_* \rho_* \pi'_* \left( \widehat{c}_1(\tau^*(\overline{\mathcal{L}}))^{\dim Z+1} \cdot \mathcal{Z} \right)$ . Then,

$$\begin{aligned} h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}_0}(Z) &= \frac{\widehat{\deg} \left( \overline{D} \cdot \left( \sum_{l=1}^d \widehat{c}_1(p_l^*(\overline{M}_l)) \right)^{\cdot d} \right)}{[K' : K](\dim Z + 1) \deg_L(Z)} \\ &= \sum_{a_1 + \dots + a_d = d} \frac{d!}{a_1! \dots a_d!} \frac{\widehat{\deg} \left( \overline{D} \cdot \prod_{l=1}^d \widehat{c}_1(p_l^*(\overline{M}_l))^{\cdot a_l} \right)}{[K' : K](\dim Z + 1) \deg_L(Z)}. \end{aligned}$$

Here we claim the following.

**Claim 2.1.1.** *If  $(a_1, \dots, a_d) \neq (1, \dots, 1)$  and  $\widehat{\deg} \left( \overline{D} \cdot \prod_{l=1}^d \widehat{c}_1(p_l^*(\overline{M}_l))^{\cdot a_l} \right) \neq 0$ , then there are  $i, j \in \{1, \dots, d\}$  such that  $a_i = 2$ ,  $a_j = 0$  and  $a_l = 1$  for all  $l \neq i, j$ . In particular,*

$$h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}_0}(Z) = d! h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}_1}(Z) + \frac{d!}{2} \sum_{i \neq j} \frac{\widehat{\deg} \left( \overline{D} \cdot \widehat{c}_1(p_i^*(\overline{M}_i))^{\cdot 2} \prod_{l=1, l \neq i, j}^d \widehat{c}_1(p_l^*(\overline{M}_l))^{\cdot a_l} \right)}{[K' : K](\dim Z + 1) \deg_L(Z)}.$$

Clearly,  $a_l \leq 2$  for all  $l$ . Thus, there is  $i$  with  $a_i = 2$ . Suppose that  $a_j = 2$  for some  $j \neq i$ . Then,

$$\widehat{\deg} \left( \overline{D} \cdot \prod_{l=1}^d \widehat{c}_1(p_l^*(\overline{M}_l))^{\cdot a_l} \right) = \widehat{\deg} \left( \overline{D} \cdot \widehat{c}_1(p_i^*(\overline{M}_i))^{\cdot 2} \cdot \widehat{c}_1(p_j^*(\overline{M}_j))^{\cdot 2} \cdot \prod_{l=1, l \neq i, j}^d \widehat{c}_1(p_l^*(\overline{M}_l))^{\cdot a_l} \right).$$

Thus, using the projection formula with respect to  $p_i$ ,

$$\widehat{\deg} \left( \overline{D} \cdot \prod_{l=1}^d \widehat{c}_1(p_l^*(\overline{M}_l))^{\cdot a_l} \right) = \widehat{\deg}(\widehat{c}_1(\overline{M}_i)^{\cdot 2}) \deg \left( D_{\eta_i} \cdot p_j^*(M_j)_{\eta_i}^{\cdot 2} \cdot \prod_{l=1, l \neq i, j}^d p_l^*(\overline{M}_l)_{\eta_i}^{\cdot a_l} \right),$$

where  $\eta_i$  means the restriction to the generic fiber of  $p_i$ . Here the generic fiber of  $p_i$  is isomorphic to  $(B_1 \times_k K_i) \times_{K_i} \dots (B_{i-1} \times_k K_i) \times_{K_i} (B_{i+1} \times_k K_i) \times (B_d \times_k K_i)$  and  $B_j \times_k K_i$  is a projective curve over  $K_i$ . Thus, we can see

$$\widehat{\deg} \left( \overline{D} \cdot \prod_{l=1}^d \widehat{c}_1(p_l^*(\overline{M}_l))^{\cdot a_l} \right) = 0.$$

This is a contradiction. Hence, we get our claim.

By the above claim, it is sufficient to see that

$$(2.1.2) \quad h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}_{i,j}}(Z) = \frac{\widehat{\deg} \left( \overline{D} \cdot \widehat{c}_1(p_i^*(\overline{M}_i))^{\cdot 2} \prod_{l=1, l \neq i, j}^d \widehat{c}_1(p_l^*(\overline{M}_l))^{\cdot a_l} \right)}{[K' : K](\dim Z + 1) \deg_L(Z)}.$$

First of all,

$$h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}_{i,j}}(Z) = \frac{-\log(\lambda_i) \int_{\mathcal{Z}(\mathbb{C})} c_1(\tau^* \overline{\mathcal{L}})^{\wedge \dim Z+1} \wedge \bigwedge_{l=1, l \neq j} c_1(\pi'^* \rho^* \nu^* p_l^*(\overline{M}_l))}{[K' : K](\dim Z + 1) \deg_L(Z)}.$$

Moreover,

$$\int_{Z(\mathbb{C})} c_1(\tau^* \overline{\mathcal{L}})^{\wedge \dim Z + 1} \wedge \bigwedge_{l=1, l \neq j} c_1(\pi'^* \rho^* \nu^* p_l^*(\overline{M}_l))$$

is equal to

$$[k : \mathbb{Q}] \deg \left( \tau^* \mathcal{L}_k^{\dim Z + 1} \cdot \prod_{l=1, l \neq j} \pi'^* \rho^* \nu^* p_l^*(\overline{M}_l)_k \cdot Z_k \right) = [k : \mathbb{Q}] \deg \left( D_k \cdot \prod_{l=1, l \neq j} p_l^*(\overline{M}_l)_k \right).$$

Thus, we obtain

$$h_{(\mathcal{X}, \overline{\mathcal{L}})}^{\overline{B}_{i,j}}(Z) = \frac{\widehat{\deg}(\widehat{c}_1(\overline{M}_i)^2) \deg \left( D_k \cdot \prod_{l=1, l \neq j} p_l^*(\overline{M}_l)_k \right)}{\deg((M_i)_k)[K' : K](\dim Z + 1) \deg_L(Z)}.$$

On the other hand, by the projection formula with respect to  $p_i$ ,

$$\widehat{\deg} \left( \overline{D} \cdot \widehat{c}_1(p_i^*(\overline{M}_i))^2 \prod_{l=1, l \neq i, j}^d \widehat{c}_1(p_l^*(\overline{M}_l)) \right) = \widehat{\deg}(\widehat{c}_1(\overline{M}_i)^2) \cdot \deg \left( D_{\eta_i} \prod_{l=1, l \neq i, j}^d p_l^*(\overline{M}_l)_{\eta_i} \right),$$

where  $\eta_i$  means the restriction to the generic fiber of  $p_i$ . Moreover, by the projection formula again,

$$\deg \left( D_k \cdot \prod_{l=1, l \neq j} p_l^*(\overline{M}_l)_k \right) = \deg(M_i)_k \cdot \deg \left( D_{\eta_i} \prod_{l=1, l \neq i, j}^d p_l^*(\overline{M}_l)_{\eta_i} \right).$$

Thus, we get (2.1.2).

The last assertion is obvious because there is a positive integer  $m$  such that

$$\overline{H}_j^{\otimes m} \succsim (\mathcal{O}_B, \lambda_i | \cdot |_{can})$$

for every  $i, j$ . □

**Corollary 2.2.** *Let  $X$  be a geometrically irreducible projective variety over  $K$ , and  $L$  an ample line bundle on  $X$ . Let  $\overline{B}$  and  $\overline{B}'$  be polarizations of  $K$ . We assume that  $\overline{B}$  is big and  $\overline{B}'$  is fairly large. Then, there are positive constants  $a$  and  $b$  such that*

$$ah_L^{\overline{B}'}(Z) + O(1) \leq h_L^{\overline{B}}(x) \leq bh_L^{\overline{B}'}(Z) + O(1)$$

for all effective cycles  $Z$  on  $X_{\overline{K}}$ . In particular, if  $X$  is an abelian variety and  $L$  is a symmetric ample line bundle, then

$$a\hat{h}_L^{\overline{B}'}(Z) \leq \hat{h}_L^{\overline{B}}(Z) \leq b\hat{h}_L^{\overline{B}'}(Z)$$

for all effective cycles  $Z$  on  $X_{\overline{K}}$ .

*Proof.* The first inequality is a consequence of [5, (5) of Proposition 3.3.7]. We set  $\overline{B} = (B; \overline{H}_1, \dots, \overline{H}_d)$  and  $\overline{B}' = (B'; \overline{H}'_1, \dots, \overline{H}'_d)$ . Since  $\overline{B}'$  is fairly large, there are generically finite morphisms  $\mu' : B'' \rightarrow B'$  and  $\nu : B'' \rightarrow (\mathbb{P}_{\mathbb{Z}}^1)^d$  of flat and projective integral schemes over  $\mathbb{Z}$ , and nef and big  $C^\infty$ -hermitian  $\mathbb{Q}$ -line bundles  $\overline{L}_1, \dots, \overline{L}_d$  on  $\mathbb{P}_{\mathbb{Z}}^1$  such that  $\mu'^*(\overline{H}'_i) \succsim \nu^*(p_i^*(\overline{L}_i))$  for all  $i$ , where  $p_i : (\mathbb{P}_{\mathbb{Z}}^1)^d \rightarrow \mathbb{P}_{\mathbb{Z}}^1$  is the projection to the  $i$ -th factor. Changing  $B''$  if necessarily, we may

assume that there is a generically finite morphism  $\mu : B'' \rightarrow B$ . By virtue of the projection formula, we may assume that  $B = B' = B''$ . We set  $\overline{H} = \nu^* \left( \bigotimes_{l=1}^d p_l^*(\overline{L}_l) \right)$ . Then,  $(B; \overline{H}, \dots, \overline{H})$  is a big polarization. Thus, there is a positive integer  $b_1$  such that

$$h_L^{\overline{B}} \leq b_1 h_L^{(B; \overline{H}, \dots, \overline{H})} + O(1).$$

Moreover, by Proposition 2.1, we can find a positive constant  $b_2$  with

$$h_L^{(B; \overline{H}, \dots, \overline{H})} \leq b_2 h_L^{(B; \nu^* p_1^*(\overline{L}_1), \dots, \nu^* p_d^*(\overline{L}_d))} + O(1).$$

On the other hand, since  $\overline{H}'_i \lesssim \nu^*(p_i^*(\overline{L}_i))$  for all  $i$ ,

$$h_L^{(B; \nu^* p_1^*(\overline{L}_1), \dots, \nu^* p_d^*(\overline{L}_d))} \leq h_L^{(B; \overline{H}_1, \dots, \overline{H}_d)} + O(1).$$

Hence, we get our corollary.  $\square$

Here, let us give the proof of Theorem I, Theorem II and Theorem III in the introduction. Theorem I is obvious by [5, Theorem 4.3] and Corollary 2.2. Theorem II is a consequence of [6], Corollary 2.2 and the following lemma.

**Lemma 2.3.** *Let  $V$  be a vector space over  $\mathbb{R}$ , and  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot, \cdot \rangle'$  be two inner products on  $V$ . If  $\langle x, x \rangle \leq \langle x, x \rangle'$  for all  $x \in V$ , then  $\det(\langle x_i, x_j \rangle) \leq \det(\langle x_i, x_j \rangle')$  for all  $x_1, \dots, x_n \in V$ .*

*Proof.* If  $x_1, \dots, x_n$  are linearly dependent, then our assertion is trivial. Otherwise, it is nothing more than [4, Lemma 3.4].  $\square$

Finally, let us consider the proof of Theorem III. The equivalence of (1) and (2) follows from Theorem II. It is obvious that (1) implies (3). Conversely, (3) implies (1) by virtue of [7] and Corollary 2.2.

### 3. NORTHCOTT'S THEOREM FOR CYCLES

In this section, we will generalize Northcott's theorem to the height of cycles on projective varieties.

**Theorem 3.1.** *Let  $K$  be a finitely generated field over  $\mathbb{Q}$ ,  $\overline{K}$  the algebraic closure of  $K$ , and let  $\overline{B}$  be a fairly large polarization of  $K$ . Let  $X$  be a geometrically irreducible projective variety over  $K$ , and  $L$  an ample line bundle on  $X$ . For an effective cycle  $Z$  on  $X_{\overline{K}}$ , we denote by  $h_L^{\overline{B}}(Z)$  the height of  $Z$  with respect to  $\overline{B}$ . Moreover, the orbit of  $Z$  by the action of the Galois group  $\text{Gal}(\overline{K}/K)$  is denoted by  $O_{\text{Gal}(\overline{K}/K)}(Z)$ . Then, for a real number  $M$  and integers  $l$  and  $e$ , the set of all effective cycles on  $X_{\overline{K}}$  with  $h_L^{\overline{B}}(Z) \leq M$ ,  $\deg_L(Z) \leq l$  and  $\#O_{\text{Gal}(\overline{K}/K)}(Z) \leq e$  is finite.*

*Proof.* Let us begin with the following lemma.

**Lemma 3.2.** *Let  $X = \mathbb{P}_{\mathbb{C}}^{n_1} \times \dots \times \mathbb{P}_{\mathbb{C}}^{n_r}$  be a product of projective spaces and  $\mathcal{O}(d_1, \dots, d_r)$  the line bundle on  $X$  with the multi-degree  $(d_1, \dots, d_r)$ . Let  $\|\cdot\|$  be the hermitian metric of  $\mathcal{O}(d_1, \dots, d_r)$  given by the Fubini-Study metric. For each  $i$ , we fix a basis of  $H^0(\mathbb{P}_{\mathbb{C}}^{n_i}, \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^{n_i}}(1))$ . Then  $H^0(X, \mathcal{O}(d_1, \dots, d_r))$  is naturally isomorphic to the space of homogeneous polynomials with the multi-degree  $(d_1, \dots, d_r)$ . For  $s \in H^0(X, \mathcal{O}(d_1, \dots, d_r))$ , we denote by  $|s|$  the maximal*

value of the absolute of coefficient of  $s$  as a polynomial. Then, there is a constant  $C$  depending only on  $n_1, \dots, n_r, d_1, \dots, d_r$  and a basis of  $H^0(\mathbb{P}_{\mathbb{C}}^{n_i}, \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^{n_i}}(1))$  for each  $i$  such that

$$|s| \leq C \exp \left( \int_X \|s\| \bigwedge_{j=1}^r p_j^*(c_1(\mathcal{O}(1), \|\cdot\|_{FS}))^{\wedge n_j} \right)$$

for all  $s \in H^0(X, \mathcal{O}(d_1, \dots, d_r))$ .

*Proof.* By virtue of [2, Corollary 1.4.3],

$$\sup_{x \in X} \{\|s\|(x)\} \leq \exp \left( \sum_{i=1}^r \sum_{m=1}^{n_i} \frac{d_i}{2m} \right) \cdot \exp \left( \int_X \|s\| \bigwedge_{j=1}^r p_j^*(c_1(\mathcal{O}(1), \|\cdot\|_{FS}))^{\wedge n_j} \right).$$

On the other hand,  $\sup_{x \in X} \{\|s\|(x)\}$  and  $|s|$  give rise to two norms on the finite dimensional space  $H^0(X, \mathcal{O}(d_1, \dots, d_r))$ . Thus, we get our lemma.  $\square$

Let us start the proof of Northcott's theorem for cycles. It is sufficient to see that, for any real number  $M$  and any integers  $e, d$  and  $l$ , the set

$$\left\{ Z \mid \begin{array}{l} Z \text{ is an effective cycle with} \\ \#O_{\text{Gal}(\overline{K}/K)}(Z) \leq e, \deg_L(Z) = d, \dim Z = l \text{ and } h_L^{\overline{B}}(Z) \leq M \end{array} \right\}$$

is finite. Clearly, we may assume that  $X = \mathbb{P}_K^n$  and  $L = \mathcal{O}_{\mathbb{P}_K^n}(1)$ . Let  $K'$  be the invariant field of the stabilizer at  $Z$ . Then,  $[K' : K] \leq e$  and  $Z$  is defined over  $K'$ . Let  $B'$  be the normalization of  $B$  in  $K'$ , and let  $\overline{H}'_1, \dots, \overline{H}'_d$  be the pull-backs of  $\overline{H}_1, \dots, \overline{H}_d$  by  $B' \rightarrow B$  respectively. Let  $\mathcal{Z}$  be the Zariski closure of  $Z$  in  $\mathbb{P}_{B'}^n = \mathbb{P}_{\mathbb{Z}}^n \times B'$ . Then,

$$h_L^{\overline{B}}(Z) = \frac{\widehat{\deg} \left( \widehat{c}_1(\overline{\mathcal{O}_{\mathbb{P}_{B'}^n}(1)})^{l+1} \cdot \prod_{j=1}^d \widehat{c}_1(\pi_{B'}^*(\overline{H}'_j)) \cdot \mathcal{Z} \right)}{[K' : K](l+1) \deg_L(Z)} + O(1),$$

where  $\pi_{B'}$  is the canonical projection  $\mathbb{P}_{B'}^n \rightarrow B'$  and  $\overline{\mathcal{O}_{\mathbb{P}_{B'}^n}(1)}$  is the pull-back of  $(\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n}(1), \|\cdot\|_{FS})$  via  $\mathbb{P}_{B'}^n \rightarrow \mathbb{P}_{\mathbb{Z}}^n$ .

Let  $\check{\mathbb{P}}_{\mathbb{Z}}^n$  be the dual projective space of  $\mathbb{P}_{\mathbb{Z}}^n$ . Let us consider

$$(\check{\mathbb{P}}_{B'}^n)^{\times_{B'}(l+1)} = \overbrace{\check{\mathbb{P}}_{B'}^n \times_{B'} \cdots \times_{B'} \check{\mathbb{P}}_{B'}^n}^{l+1},$$

where  $\check{\mathbb{P}}_{B'}^n = \check{\mathbb{P}}_{\mathbb{Z}}^n \times B'$ . Let  $p_i : (\check{\mathbb{P}}_{B'}^n)^{\times_{B'}(l+1)} \rightarrow \check{\mathbb{P}}_{B'}^n$  be the projection to the  $i$ -th factor and  $p : (\check{\mathbb{P}}_{B'}^n)^{\times_{B'}(l+1)} \rightarrow B'$  the canonical morphism. We set

$$\mathcal{O}_{B'}(d, \dots, d) = \bigotimes_{i=1}^{l+1} p_i^*(\mathcal{O}_{\check{\mathbb{P}}_{B'}^n}(1)).$$



Let  $\text{Ch}(Z)$  be the Chow divisor of  $Z$ , i.e.,  $\text{Ch}(Z)$  is an element of  $|\mathcal{O}_{K'}(d, \dots, d)|$  on  $(\check{\mathbb{P}}_{K'}^n)^{l+1}$ . Let  $\text{Ch}(\mathcal{Z})$  be the Zariski closure of  $\text{Ch}(Z)$  in  $(\check{\mathbb{P}}_{B'}^n)^{\times_{B'}(l+1)}$ . Here we claim the following equation:

$$(3.2.1) \quad p_* \left( \prod_{i=1}^{l+1} \widehat{c}_1 \left( p_i^* (\overline{\mathcal{O}_{\check{\mathbb{P}}_{B'}^n}(1)}) \right) \right)^n \cdot \text{Ch}(\mathcal{Z}) \\ = \pi_* \left( \widehat{c}_1 (\overline{\mathcal{O}_{\check{\mathbb{P}}_{B'}^n}(1)})^{l+1} \cdot \mathcal{Z} \right) + d(l+1) \pi_* \left( \widehat{c}_1 (\overline{\mathcal{O}_{\check{\mathbb{P}}_{B'}^n}(1)})^{n+1} \right).$$

Let  $U$  be the maximal Zariski open set of  $B'$  such that  $\mathcal{Z} \rightarrow B'$  is flat over  $U$ . Then, in the same way as in the proof of [1, Proposition 1.2], we can see that the equation (3.2.1) holds over  $U$ . Therefore, so does over  $B'$  by [3, Lemma 2.5.1] because  $\text{codim}(B' \setminus U) \geq 2$ .

By using (3.2.1), we can see

$$(3.2.2) \quad \left( h_L^{\overline{B}}(Z) + O(1) \right) + \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{O}_{\check{\mathbb{P}}_{\mathbb{Z}}^n}(1)})) \deg(H_{1\mathbb{Q}} \cdots H_{d\mathbb{Q}}) \\ = \frac{\widehat{\deg} \left( \prod_{i=1}^{l+1} \widehat{c}_1 \left( p_i^* (\overline{\mathcal{O}_{\check{\mathbb{P}}_{B'}^n}(1)}) \right) \right)^n \cdot \prod_{j=1}^d \widehat{c}_1(p^*(\overline{H'_j})) \cdot \text{Ch}(\mathcal{Z})}{[K' : K]}.$$

Choose  $P \in H^0 \left( (\check{\mathbb{P}}_{K'}^n)^{l+1}, \mathcal{O}_{K'}(d, \dots, d) \right)$  with  $\text{div}(P) = \text{Ch}(Z)$ . Here, we fix a basis of  $H^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n}(1))$ . Then,  $P$  can be written by a polynomial with coefficients  $\{a_\lambda\}_{\lambda \in \Lambda}$  in  $K'$ , that is,  $\{a_\lambda\}_{\lambda \in \Lambda}$  is a Chow coordinate of  $Z$ . Noting that  $P$  gives rise a rational section of  $\mathcal{O}_{B'}(d, \dots, d)$  over  $B'$ , let

$$\text{div}(P) = \text{Ch}(\mathcal{Z}) + \sum_{\Gamma} c_{\Gamma} p^*(\Gamma)$$

be the decomposition as a rational section of  $\mathcal{O}_{B'}(d, \dots, d)$ , where  $\Gamma$  runs over all prime divisors on  $B'$ . Then, we can easily see  $c_{\Gamma} = \min_{\lambda \in \Lambda} \{\text{ord}_{\Gamma}(a_{\lambda})\}$ . Here, let us calculate

$$\widehat{\deg} \left( \prod_{i=1}^{l+1} \widehat{c}_1 \left( p_i^* (\overline{\mathcal{O}_{\check{\mathbb{P}}_{B'}^n}(1)}) \right) \right)^n \cdot \prod_{j=1}^d \widehat{c}_1(p^*(\overline{H'_j})) \cdot \widehat{c}_1(\overline{\mathcal{O}_{B'}(d, \dots, d)})$$

in terms of the rational section  $P$ . Then,

$$(3.2.3) \quad \widehat{\deg} \left( \prod_{i=1}^{l+1} \widehat{c}_1 \left( p_i^* (\overline{\mathcal{O}_{\check{\mathbb{P}}_{B'}^n}(1)}) \right) \right)^n \cdot \prod_{j=1}^d \widehat{c}_1(p^*(\overline{H'_j})) \cdot \text{Ch}(\mathcal{Z}) = \\ [K' : K] \widehat{\deg} \left( \prod_{i=1}^{l+1} \widehat{c}_1 \left( p_i^* (\overline{\mathcal{O}_{\check{\mathbb{P}}_B^n}(1)}) \right) \right)^n \cdot \prod_{j=1}^d \widehat{c}_1(p^*(\overline{H_j})) \cdot \widehat{c}_1(\overline{\mathcal{O}_B(d, \dots, d)}) \\ + \sum_{\Gamma} \max_{\lambda \in \Lambda} \{-\text{ord}_{\Gamma}(a_{\lambda})\} \widehat{\deg} \left( \prod_{j=1}^d \widehat{c}_1(\overline{H'_j}) \cdot \Gamma \right) \\ + \int_{(\check{\mathbb{P}}_{\mathbb{C}}^n)^{l+1} \times_{B'(\mathbb{C})}} \log \|P\| \bigwedge_{i=1}^{l+1} c_1 \left( p_j^* (\overline{\mathcal{O}_{\check{\mathbb{P}}_{B'}^n}(1)}) \right)^{\wedge n} \wedge \bigwedge_{j=1}^d c_1(p^*(\overline{H'_j})).$$

Let  $U_0$  be a Zariski open set of  $B'(\mathbb{C})$  such that  $a_\lambda$  has no zeros and poles on  $U_0$  for every  $\lambda \in \Lambda$ . For each  $b \in U_0$ , let  $i_b : (\check{\mathbb{P}}_{\mathbb{C}}^n)^{l+1} \rightarrow (\check{\mathbb{P}}_{\mathbb{C}}^n)^{l+1} \times B'(\mathbb{C})$  be a morphism given by  $i_b(x) = (x, b)$ . Then,

$$\begin{aligned} \left( \int_{(\check{\mathbb{P}}_{\mathbb{C}}^n)^{l+1} \times B'(\mathbb{C})} \log \|P\| \bigwedge_{i=1}^{l+1} c_1 \left( p_j^* (\overline{\mathcal{O}_{\check{\mathbb{P}}_{B'}^n}(1)}) \right)^{\wedge n} \right) (b) \\ = \int_{(\check{\mathbb{P}}_{\mathbb{C}}^n)^{l+1}} i_b^* (\log \|P\|) \bigwedge_{i=1}^{l+1} c_1 \left( p_j^* (\overline{\mathcal{O}_{\check{\mathbb{P}}_{\mathbb{C}}^n}(1)}) \right)^{\wedge n} \end{aligned}$$

Thus, by Lemma 3.2,

$$\begin{aligned} (3.2.4) \quad \int_{(\check{\mathbb{P}}_{\mathbb{C}}^n)^{l+1} \times B'(\mathbb{C})} \log \|P\| \bigwedge_{i=1}^{l+1} c_1 \left( p_j^* (\overline{\mathcal{O}_{\check{\mathbb{P}}_{B'}^n}(1)}) \right)^{\wedge n} \wedge \bigwedge_{j=1}^d c_1(p^*(\overline{H}_j)) \\ \geq \int_{B'(\mathbb{C})} \log \max_{\lambda \in \Lambda} \{|a_\lambda|\} \bigwedge_{j=1}^d c_1(\overline{H}_j) - [K' : K] C' \int_{B(\mathbb{C})} \bigwedge_{j=1}^d c_1(\overline{H}_j) \end{aligned}$$

for some constant  $C'$  depending only on  $n, d, l$  and a basis  $H^0(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n}(1))$ . Thus, gathering (3.2.2), (3.2.3) and (3.2.4),

$$h_{nv}^{\overline{B}}((a_\lambda)_{\lambda \in \Lambda}) \leq h_L^{\overline{B}}(Z) + C''$$

for some constant  $C''$  independent on  $Z$ . Thus, we have only finitely many  $(a_\lambda)_{\lambda \in \Lambda}$  modulo the scalar product of  $\overline{K}$ . Therefore, we have only finitely many  $\text{Ch}(Z)$ . Hence we obtain our assertion because the correspondence

$$\text{Ch} : \left\{ \begin{array}{l} \text{Effective cycles } Z \text{ on } X_{\overline{K}} \text{ with} \\ l = \dim Z \text{ and } \deg(Z) = d \end{array} \right\} \rightarrow |\mathcal{O}_{\overline{K}}(d, \dots, d)|$$

is injective.  $\square$

#### Appendix. DIRECT PROOF OF NORTHCOTT'S THEOREM WITH RESPECT TO A FAIRLY LARGE POLARIZATION

If we use a fairly large polarization, we can give a simpler proof of Northcott's theorem. In this appendix, let us consider this problem.

Let us start the direct proof of Theorem I. We denote by  $\overline{\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^1}(1)}$  the hermitian line bundle  $(\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^1}(1), \|\cdot\|_{FS})$  on  $\mathbb{P}_{\mathbb{Z}}^1$ . First of all, we claim the following.

**Claim A.1.** *We assume that there is a generically finite morphism  $\nu : B \rightarrow (\mathbb{P}_{\mathbb{Z}}^1)^d$  with  $\overline{H}_i = \nu^*(p_i^*(\overline{\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^1}(1)}))$  for all  $i$ . Then, the assertion of our theorem holds.*

Since  $L$  is ample, there is a positive integer  $m$  and an embedding  $\phi : X \hookrightarrow \mathbb{P}^n$  with  $\phi^*(\mathcal{O}_{\mathbb{P}^n}(1)) = L^{\otimes m}$ . Thus, we may assume that  $X = \mathbb{P}_K^n$  and  $L = \mathcal{O}_{\mathbb{P}_K^n}(1)$ . Let  $\overline{B}_0$  be a polarization given by  $((\mathbb{P}_{\mathbb{Z}}^1)^d; p_1^*(\overline{\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^1}(1)}), \dots, p_d^*(\overline{\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^1}(1)}))$ . Then, by the projection formula, we can see that  $h_L^{\overline{B}_0} = \deg(\nu) h_L^{\overline{B}} + O(1)$ . Therefore, we may assume  $\overline{B} = \overline{B}_0$ . Moreover, in the same argument as in [5, Claim 4.3.3], we may assume  $e = 1$ .

Let  $\Delta_\infty$  be the closure of  $\infty \in \mathbb{P}_\mathbb{Q}^1$  in  $\mathbb{P}_\mathbb{Z}^1$ . We set  $\Delta_\infty^{(i)} = p_i^*(\Delta_\infty)$ . Moreover, we set  $\overline{A}_i = p_i^*(\mathcal{O}_{\mathbb{P}_\mathbb{Z}^1}, (1/2)| \cdot |_{{\text{can}}})$ . If we denote  $c_1(\mathbb{P}^1, \| \cdot \|_{FS})$  by  $\omega$ , then

(A.2)

$$\widehat{\deg}(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{A}_i) \cdots \widehat{c}_1(\overline{H}_d) \cdot \Delta_\infty^{(j)}) = \int_{p_j^{-1}(\infty)} \log(2) \bigwedge_{l=1, l \neq i}^d p_l^*(\omega) = \begin{cases} \log(2) & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

Let  $\overline{B}_i$  be a polarization of  $K$  given by  $\overline{B}_i = (B; \overline{H}_1, \dots, \overline{H}_{i-1}, \overline{A}_i, \overline{H}_{i+1}, \dots, \overline{H}_d)$ . Here  $\overline{\mathcal{O}_{\mathbb{P}_\mathbb{Z}^1}(1)}^{\otimes 2} \succ (\mathcal{O}_{\mathbb{P}_\mathbb{Z}^1}, (1/2)| \cdot |_{{\text{can}}})$  because  $\sup_{x \in \mathbb{P}^1(\mathbb{C})} \|X_0 X_1\|_{FS}(x) \leq 1/2$ , where  $\{X_0, X_1\}$  are a basis of  $H^0(\mathbb{P}_\mathbb{Z}^1, \mathcal{O}_{\mathbb{P}_\mathbb{Z}^1}(1))$ . Thus, by [5, (5) of Proposition 3.3.7], there are positive constants  $a$  such that  $h_{nv}^{\overline{B}_i} \leq 2h_{nv}^{\overline{B}} + a$  for all  $i$ . We set

$$S = \{P \in \mathbb{P}^n(\mathbb{Q}(z_1, \dots, z_d)) \mid h_{nv}^{\overline{B}}(P) \leq M\}$$

Then, for any  $P \in S$ ,  $h_{nv}^{\overline{B}_i}(P) \leq 2M + a$ . Moreover, there are  $f_0, \dots, f_n \in \mathbb{Z}[z_1, \dots, z_d]$  such that  $f_0, \dots, f_n$  are relatively prime and  $P = (f_0 : \dots : f_n)$ . Here, by using (A.2) together with facts that  $c_1(\overline{A}_i) = 0$  and  $f_0, \dots, f_n$  are relatively prime,

$$h_{nv}^{\overline{B}_i}(P) = \max\{\deg_i(f_0), \dots, \deg_i(f_n)\} \log(2),$$

where  $\deg_i$  is the degree of polynomials with respect to  $z_i$ . Thus, there is a constant  $M_1$  independent on  $P \in S$  such that  $\deg_i(f_j) \leq M_1$  for all  $i, j$ . On the other hand,

$$\begin{aligned} h_{nv}^{\overline{B}}(P) &= \sum_i \max\{\deg_i(f_0), \dots, \deg_i(f_n)\} \widehat{\deg}(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d) \cdot \Delta_\infty^{(i)}) \\ &\quad + \int_{(\mathbb{P}^1)^d} \log\left(\max_i \{|f_i|\}\right) c_1(\overline{H}_1) \wedge \cdots \wedge c_1(\overline{H}_d) \end{aligned}$$

Hence, there is a constant  $M_2$  independent on  $P$  such that

$$\int_{(\mathbb{P}^1)^d} \log(|f_i|) c_1(\overline{H}_1) \wedge \cdots \wedge c_1(\overline{H}_d) \leq M_2$$

for all  $i$ . Thus, by [5, Lemma 4.1], we have our claim.

Let us consider a general case. We use the notation in the definition of the largeness of a polarization. Clearly, we may assume that  $X = \mathbb{P}_K^n$  and  $L = \mathcal{O}_{\mathbb{P}_K^n}(1)$ . Let  $K'$  be the function field of  $B'$ , and  $\overline{B}'$  a polarization of  $K'$  given by  $(B'; \mu^*(H_1), \dots, \mu^*(H_d))$ . Then, for all  $x \in \mathbb{P}^n(\overline{K})$ ,

$$h_{L_{K'}}^{\overline{B}'}(x) = \frac{1}{[K' : K]} h_L^{\overline{B}}(x).$$

Thus, we may also assume that  $B' = B$ . Moreover, there is a positive integer  $b$  with  $\overline{L}_i^{\otimes b} \succ \overline{\mathcal{O}_{\mathbb{P}_\mathbb{Z}^1}(1)}$  for every  $i$ . Hence,  $\overline{H}_i^{\otimes b} \succ \nu^*(p_i^*(\overline{\mathcal{O}_{\mathbb{P}_\mathbb{Z}^1}(1)}))$ . Let  $\overline{B}'$  be a polarization of  $K$  given by

$$\overline{B}' = \left(B; \nu^*(p_1^*(\overline{\mathcal{O}_{\mathbb{P}_\mathbb{Z}^1}(1)})), \dots, \nu^*(p_d^*(\overline{\mathcal{O}_{\mathbb{P}_\mathbb{Z}^1}(1)}))\right).$$

Let  $\overline{\mathcal{O}_{\mathbb{P}_B^n}(1)}$  be a  $C^\infty$ -hermitian line bundle on  $\mathbb{P}_B^n$  given the pull-back of  $(\mathcal{O}_{\mathbb{P}_Z^n}(1), \|\cdot\|_{FS})$  via  $\mathbb{P}_B^n \rightarrow \mathbb{P}_Z^n$ . Then, since  $\overline{\mathcal{O}_{\mathbb{P}_Z^1}(1)}$  and  $\overline{\mathcal{O}_{\mathbb{P}_B^n}(1)}$  are nef, we can see that, for all  $x \in \mathbb{P}^n(\overline{K})$ ,

$$h_{(\mathbb{P}_B^n, \overline{\mathcal{O}_{\mathbb{P}_B^n}(1)})}^{\overline{B}'}(x) \leq b^d h_{(\mathbb{P}_B^n, \overline{\mathcal{O}_{\mathbb{P}_B^n}(1)})}^{\overline{B}}(x).$$

Thus, by the previous claim, we get our theorem.  $\square$

**Remark A.3.** In order to guarantee Northcott's theorem, the largeness of a polarization is crucial. The following example shows us that even if the polarization is ample in the geometric sense, Northcott's theorem does not hold.

Let  $k = \mathbb{Q}(\sqrt{29})$ ,  $\epsilon = (5 + \sqrt{29})/2$ , and  $O_k = \mathbb{Z}[\epsilon]$ . We set

$$E = \text{Proj} (O_k[X, Y, Z]/(Y^2Z + XYZ + \epsilon^2YZ^2 - X^3)).$$

Then,  $E$  is an abelian scheme over  $O_k$ . Then, as in the proof of [5, Proposition 3.1.1], we can construct a nef  $C^\infty$ -hermitian line bundle  $\overline{H}$  on  $E$  such that  $[2]^*(\overline{H}) = \overline{H}^{\otimes 4}$  and  $H_k$  is ample on  $E_k$ ,  $c_1(\overline{H})$  is positive on  $E(\mathbb{C})$ , and that  $\widehat{\deg}(\widehat{c_1}(\overline{H})^2) = 0$ . Let  $K$  be the function field of  $E$ . Then,  $\overline{B} = (E; \overline{H})$  is a polarization of  $K$ . Here we claim that Northcott's theorem does not hold for the polarization  $(E, \overline{H})$  of  $K$ .

Let  $p_i : E \times_{O_k} E \rightarrow E$  be the projection to the  $i$ -th factor. Then, considering  $p_2 : E \times_{O_k} E \rightarrow E$ ,  $(E \times_{O_k} E, p_1^*(\overline{H}))$  gives rise to a model of  $(E_K, H_K)$ . Let  $\Gamma_n$  be the graph of  $[2]^n : E \rightarrow E$ , i.e.,  $\Gamma_n = \{([2]^n(x), x) \mid x \in E\}$ . Moreover, let  $x_n$  be a  $K$ -valued point of  $E_K$  arising from  $\Gamma_n$ . Then, if we denote the section  $E \rightarrow \Gamma_n$  by  $s_n$ , then

$$\begin{aligned} h_{H_K}^{\overline{B}}(x_n) &= \widehat{\deg}(p_1^*(\overline{H}) \cdot p_2^*(\overline{H}) \cdot \Gamma_n) = \widehat{\deg}(s_n^*(p_1^*(\overline{H})) \cdot s_n^*(p_2^*(\overline{H}))) \\ &= \widehat{\deg}([2]^n)^*(\overline{H}) \cdot \overline{H}) = \widehat{\deg}(\overline{H}^{\otimes 4^n} \cdot \overline{H}) = 4^n \widehat{\deg}(\overline{H} \cdot \overline{H}) = 0. \end{aligned}$$

On the other hand,  $x_n$ 's are distinct points in  $E_K(K)$ .

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